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A HOLLOW ELLIPSOIDAL NEEDLE IN AN ORTHOTROPIC ELASTIC MEDIUM*

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The problem of the stress distribution on the surface of a hollow ellipsoidal needle in an orthotropic elastic medium and a homogeneous external field is solved. Explicit expressions are obtained for the stresses on the needle surface in terms of the elastic constants of the medium and parameters of the ellipsoid in a local system of coordinates connected to the normal to the surface at each point of the needle. The general solution of the problem of the stress concentration on an ellipsoidal inhomogeneity /1/ and the passage to the limit cases of an ellipsoidal cavity based on the presence of small parameters /2/ is used.

1. Consider a hollow ellipsoidal needle, i.e., an ellipsoidal cavity, one of whose dimensions is large compared with the other two, in an orthotropic unbounded elastic medium subjected to an external uniform field $\sigma_0^{\alpha\beta}$. The equation of the ellipsoid is written in the form

$$x_1^2a_1^{-2} + x_2^2a_2^{-2} + x_3^2a_3^{-2} = 1,\ a_1 \gg a_2 \geqslant a_3 \tag{1.1}$$

in an (x_1, x_2, x_3) system of coordinates rigidly connected to the ellipsoid.

We will assume that the axes of elastic symmetry of the external orthotropic medium coincide with the axes of the ellipsoid. Then the tensor of the elastic constants of the medium $c^{\alpha\beta\lambda\mu}$ has nine non-zero components that are denoted according to the usual rule /3/ by

$$c^{\alpha\alpha\beta\beta} = c_{\alpha\beta}\ (\alpha,\ \beta = 1,\ 2,\ 3) \tag{1.2}$$
$$c^{2323} = c_{44},\ c^{3131} = c_{55},\ c^{1212} = c_{66}$$

The stresses $\sigma^{\alpha\beta}(\mathbf{n})$ on the surface of an ellipsoidal cavity in a uniform external field $\sigma_0^{\lambda\mu}$ have the form

$$\sigma^{\alpha\beta}(\mathbf{n}) = F^{\alpha\beta}_{\cdot\cdot\lambda\alpha}(\mathbf{n})\,\sigma_0^{\lambda\mu},\quad F^{\alpha\beta}_{\cdot\cdot\lambda\mu}(\mathbf{n}) = B^{\alpha\beta\varkappa\rho}(\mathbf{n})\,B^{-1}_{\varkappa\rho\lambda\mu} \tag{1.3}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the normal to the ellipsoid surface.

The tensor stress concentration coefficient $F^{\alpha\beta}_{\cdot\cdot\lambda\mu}(\mathbf{n})$ can be represented in the form of the product of two factors. The first of them, the tensor $B^{\alpha\beta\varkappa\rho}(\mathbf{n})$ depends only on the elastic constants of the medium and the inclusion and on the normal $\mathbf{n}$ to the inclusion surface, and remains finite for any passages to the limit. For a cavity the tensor $B(\mathbf{n})$ has the form /1/

$$B^{\alpha\beta\lambda\mu}(\mathbf{n}) = c^{\alpha\beta\lambda\mu} - c^{\alpha\beta\varkappa\rho}K_{\varkappa\rho\eta\nu}(\mathbf{n})\,c^{\eta\nu\lambda\mu} \tag{1.4}$$

where the tensor $K(\mathbf{n})$ for an orthotropic medium is constructed explicitly in terms of the Fourier transform of Green's tensor of a homogeneous medium. The expressions for the components of $K(\mathbf{n})$ in terms of the elastic constants of an orthotropic medium and the coordinates

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of the unit vector of the normal $\mathbf{n}$ are presented in /4/, and for an isotropic medium in /1/.

The tensor $B(\mathbf{n})$ for a cavity is symmetric within pairs and with respect to permutation of pairs of superscripts $B^{\alpha\beta\lambda\mu}(\mathbf{n}) = B^{\lambda\mu\alpha\beta}(\mathbf{n}) = B^{\beta\alpha\lambda\mu}(\mathbf{n})$ and its components are obtained from (1.4) by convolution of appropriate components of the tensors c and $K(\mathbf{n})$.

Because of the awkwardness of the expressions for the components of $B(\mathbf{n})$ we will present their values over the principal ellipsoid sections $n_1 = 0$, $n_2 = 0$, $n_3 = 0$. The non-zero components of the tensor $B(\mathbf{n})$ in the section $n_1 = 0$ have the form

$$
\begin{gathered}
B^{1111}(\mathbf{n}) = p_1[\Delta_{33}n_2^4 + \Delta_{22}n_3^4 + q_1 n_2^2 n_3^2] \\
B^{2222}(\mathbf{n}) = p_1\Delta_{11}n_3^4,\quad B^{3333}(\mathbf{n}) = p_1\Delta_{11}n_2^4,\quad B^{2233}(\mathbf{n}) = p_1\Delta_{11}n_2^2 n_3^2 \\
B^{2322}(\mathbf{n}) = -p_1\Delta_{11}n_2 n_3^3,\ B^{2333}(\mathbf{n}) = p_1\Delta_{11}n_2^3 n_3,\ B^{2323}(\mathbf{n}) = B^{2233}(\mathbf{n}) \\
B^{1122}(\mathbf{n}) = -\eta_1 n_3^2,\ B^{1133}(\mathbf{n}) = -\eta_1 n_2^2,\ B^{2311}(\mathbf{n}) = \eta_1 n_2 n_3 \\
B^{1313}(\mathbf{n}) = \psi_1 n_2^2,\ B^{1212}(\mathbf{n}) = \psi_1 n_3^2,\ B^{1213}(\mathbf{n}) = \psi_1 n_2 n_3 \\
p_1 = [c_{22}n_2^4 + c_{33}n_3^4 + (\Delta_{11}c_{44}^{-1} - 2c_{23})\,n_2^2 n_3^2]^{-1} \\
\eta_1 = p_1(\Delta_{13}n_2^2 + \Delta_{12}n_3^2),\ \psi_1 = c_{55}c_{66}(c_{66}n_2^2 + c_{55}n_3^2)^{-1} \\
q_1 = \Delta c_{44}^{-1} + 2\Delta_{23}
\end{gathered}
\tag{1.5}
$$

Here $\Delta_{\alpha\beta}$ is the cofactor of the element $c_{\alpha\beta}$ of the matrix $\|c_{\alpha\beta}\|$ ($\alpha, \beta = 1, 2, 3$) and Δ is the determinant of this matrix.

The components of the tensor $B(\mathbf{n})$ in the sections $n_2 = 0$ and $n_3 = 0$ are obtained from (1.5) by the change in subscripts $1 \leftrightarrow 2$, $4 \leftrightarrow 5$ and $1 \leftrightarrow 3$, $4 \leftrightarrow 6$, respectively. The expression for $B(\mathbf{n})$ for an isotropic medium is given in /2/.

The factor B^{-1} of the concentration coefficient $F(\mathbf{n})$ in (1.3) depends on the shape of the inhomogeneity and is a constant tensor inverse to the tensor B for an ellipsoid, which is expressed in terms of the mean value of $B(\mathbf{n})$ over the ellipsoid surface

$$
\begin{gathered}
B^{\varkappa\rho\lambda\mu} = \langle B^{\varkappa\rho\lambda\mu}(\mathbf{n})\rangle = (4\pi)^{-1} a_1 a_2 a_3 \int_\Omega B^{\varkappa\rho\lambda\mu}(\mathbf{n})\,\rho(\mathbf{n})\,d\mathbf{n} \\
\rho(\mathbf{n}) = (a_1^2 n_1^2 + a_2^2 n_2^2 + a_3^2 n_3^2)^{-3/2}
\end{gathered}
\tag{1.6}
$$

where the integration is performed over all directions of the unit vector $\mathbf{n}$, i.e., over the surface of the unit sphere. The dependence of the tensor B on the ellipsoid parameters is concentrated in the scalar weighting factor $\rho(n)$, which considerably facilitates the passage to the limit cases, and enables explicit expressions to be obtained for the stresses on the surface of needles, cracks, and discs.

2. We will use the method proposed in /2/ for the passage to the limit case of a needle. We introduce the dimensionless parameters $\zeta = a_2 a_1^{-1}$, $\xi = a_3 a_2^{-1}$. Then $\zeta \ll 1$, $\xi \sim 1$ corresponds to the needle. We expand the weighting factor $\rho(n)$ in (1.6) in the small (but finite) parameter ζ by extracting the singular component $\delta(t)$ therein and substituting the expansion of $\rho(n)$ into (1.6), we obtain an expansion of the tensor B:

$$
\begin{gathered}
B = B_0 + O(\zeta^2 \ln \zeta),\quad B_0 = \frac{\xi}{2\pi}\int_{-\pi}^{\pi} \frac{B(\varphi, 0)\,d\varphi}{\cos^2\varphi + \xi^2\sin^2\varphi} \\
B(\varphi, 0) = B(n_1 = 0,\ n_2 = \cos\varphi,\ n_3 = \sin\varphi) \\
n_1 = \cos\Theta,\ n_2 = \sin\Theta\cos\varphi,\ n_3 = \sin\Theta\sin\varphi,\ \cos\Theta = t
\end{gathered}
\tag{2.1}
$$

(φ, Θ are spherical coordinates with the Θ axis directed along the needle axis x_1).

Since the tensor B_0 for any anisotropic medium has an inverse /2/, we can limit ourselves to the principal term B_0 in the expansion (2.1). Therefore, the solution of the problem of the stress distribution on the needle surface reduces to evaluating the single integrals (2.1) and the inversion of the tensor B_0.

We obtain the principal terms of the expansion of the components of the tensors B and B^{-1} is the small parameter ζ for an orthotropic medium. We note that B and B^{-1} have the symmetry and structure of the tensor of elastic constants. We will first write the tensor $B(\varphi, 0)$. It follows from formulas (1.5) for $B(\mathbf{n})$ the the arbitrary non-zero component of the tensor $B(\varphi, 0)$ (with the exception of 1212 and 1313) can be represented in the form

$$
\begin{gathered}
B(\varphi, 0) = p_1(\varphi)\,[P\sin^4\varphi + Q\sin^2\varphi\cos^2\varphi + R\cos^4\varphi] \\
p_1(\varphi) = p_1(n_2 = \cos\varphi,\ n_3 = \sin\varphi)
\end{gathered}
\tag{2.2}
$$

The tensors P, Q and R have the symmetry of the elastic constants tensor and their non-zero components are expressed in terms of the elastic characteristics as follows

$$\begin{gathered} P^{1111} = \Delta_{22},\ R^{1111} = \Delta_{33},\ Q^{1111} = q_1 \\ P^{2222} = Q^{2233} = Q^{2323} = R^{3333} = \Delta_{11} \\ P^{1122} = Q^{1133} = -\Delta_{12},\ Q^{1122} = R^{1133} = -\Delta_{13} \end{gathered} \tag{2.3}$$

Substituting (2.2) and (2.3) for $B(\varphi, 0)$ into (2.1) and integrating over φ, we obtain

$$\begin{gathered} B = (kMc_{33})^{-1}[k(l + \xi k)P + \xi kQ + \xi(1 + l\xi)R] \\ B^{1212} = \gamma\sqrt{c_{66}},\ B^{1313} = \xi\gamma\sqrt{c_{55}},\ \gamma = \sqrt{c_{55}c_{66}}\,(\sqrt{c_{55}} + \xi\sqrt{c_{66}})^{-1} \\ k = \sqrt{\frac{c_{22}}{c_{33}}},\quad l = \sqrt{\frac{\Delta_{11} - 2c_{23}c_{44}}{c_{33}c_{44}} + 2k},\quad M = l(1 + \xi l + k\xi^2) \end{gathered} \tag{2.4}$$

The tensor B^{-1}, inverse to B, is determined by the relationships

$$B^{\alpha\beta\lambda\mu}B^{-1}_{\lambda\mu\sigma\tau} = I^{\alpha\beta}_{\cdot\cdot\sigma\tau} = {}^1/_2\,(\delta_\sigma{}^\alpha\delta_\tau{}^\beta + \delta_\tau{}^\alpha\delta_\sigma{}^\beta)$$

where $I^{\alpha\beta}_{\cdot\cdot\sigma\tau}$ is a single quadrivalent tensor symmetric within the pair $(\alpha\beta)$, $(\sigma\tau)$ and by commutation of the pairs of indices.

Construction of the tensor B^{-1} reduces to inversion of a sixth-order matrix. The components B with indices 1212, 1313, 2323 are found directly as quantities inverse to the corresponding quadruple component of the tensor B. The remaining six independent components of B^{-1} are elements of a symmetric third-order matrix $\|B_{\alpha\beta}{}^{-1}\|$ inverse to $\|B^{\alpha\beta}\|$ where

$$B^{\alpha\beta} = B^{\alpha\alpha\beta\beta},\ B_{\alpha\beta}{}^{-1} = B^{-1}_{\alpha\alpha\beta\beta}\ (\alpha,\ \beta = 1,\ 2,\ 3)$$

Carrying out the necessary calculations, we obtain

$$\begin{gathered} B^{-1}_{11\alpha\alpha} = \Delta_{1\alpha}\Delta^{-1}\ (\alpha = 1,\ 2,\ 3) \\ B^{-1}_{2222} = \Delta_{22}\Delta^{-1} + \xi L\,(k\Delta_{11})^{-1},\ B^{-1}_{3333} = \Delta_{33}\Delta^{-1} + L\,(\xi\Delta_{11})^{-1} \\ B^{-1}_{2233} = \Delta_{11}{}^{-1}\,(\Delta_{12}\Delta_{13}\Delta^{-1} - kc_{33}),\ \ B^{-1}_{2323} = c_{33}M\,(4\xi\Delta_{11})^{-1} \\ B^{-1}_{1212} = (4\gamma\sqrt{c_{66}})^{-1},\ B_{1313} = (4\xi\gamma\sqrt{c_{55}})^{-1},\ L = klc_{33} \end{gathered} \tag{2.5}$$

If we introduce Young's modulus E_α and Poisson's ratios $\nu_{\alpha\beta}$ of an orthotropic medium /5/, then

$$\Delta_{\alpha\alpha}\Delta^{-1} = E_\alpha{}^{-1},\ \ \Delta_{\alpha\beta}\Delta^{-1} = -\nu_{\alpha\beta}E_\alpha{}^{-1}\ (\alpha \neq \beta,\ \alpha,\ \beta = 1,\ 2,\ 3)$$

The expressions for the components of the tensor B^{-1} for an isotropic medium are presented in /6/.

Since $\xi \sim 1$ for the needle, the components of the tensor B^{-1} are always finite. Consequently, the concentration coefficient $F(\mathbf{n})$ tends to a finite value as $\zeta \to 0$ and, therefore, the stresses $\sigma^{\alpha\beta}(n)$ have no singularities on the needle surface.

The expressions for the components of the tensor $F(\mathbf{n})$ over the principal sections of the ellipsoid are obtained in explicit form, but are not presented here because of their awkwardness. We will present at once the expressions for the stresses on the needle surface.

We will first obtain expressions for the non-zero components of the stress tensor at the needle apices $A\,(n_1 = 1,\ n_2 = n_3 = 0)$, $B\,(n_1 = n_3 = 0,\ n_2 = 1)$, $C\,(n_1 = n_2 = 0;\ n_3 = 1)$

$$\begin{gathered} \sigma^{22}(A) = -c_{12}c_{11}{}^{-1}\sigma_0{}^{11} + (c_{11}\Delta_{11})^{-1}\{[c_{33}\Delta_{33}(1 + \xi l) - c_{12}\Delta_{12} + \\ kc_{33}\Delta_{23}]\sigma_0{}^{22} + [\Delta_{33}(c_{23} - kc_{33}) - L\Delta_{23}\xi^{-1}]\sigma_0{}^{23}\} \\ \sigma^{33}(A) = -c_{13}c_{11}{}^{-1}\sigma_0{}^{11} + (c_{11}\Delta_{11})^{-1}\{[\Delta_{22}(c_{23} - kc_{33}) - \xi lc_{33}\Delta_{23}]\sigma_0{}^{22} + \\ [c_{33}\Delta_{33} - c_{12}\Delta_{12} + L(\xi^{-1}\Delta_{22} + k^{-1}\Delta_{23})]\sigma_0{}^{33}\} \\ \sigma^{23}(A) = M(\xi\Delta_{11})^{-1}\sigma_0{}^{23},\quad \sigma^{13}(B) = \sqrt{c_{55}}\,(\xi\gamma)^{-1}\sigma_0{}^{13} \\ \sigma^{11}(B) = \sigma_0{}^{11} + \Delta_{11}{}^{-1}[(\Delta_{12} + k^{-1}\Delta_{13})\sigma_0{}^{22} - l(k\xi)^{-1}\Delta_{13}\sigma_0{}^{33}] \\ \sigma^{33}(B) = -k^{-1}\sigma_0{}^{22} + [1 + l(k\xi)^{-1}]\sigma_0{}^{33} \\ \sigma^{11}(C) = \sigma_0{}^{11} - \Delta_{11}{}^{-1}[\xi l\Delta_{12}\sigma_0{}^{22} - (\Delta_{13} + k\Delta_{12})\sigma_0{}^{33}] \\ \sigma^{22}(C) = (1 + \xi l)\sigma_0{}^{22} - k\sigma_0{}^{33},\quad \sigma^{12}(C) = \sqrt{c_{66}}\,\xi^{-1}\sigma_0{}^{12} \end{gathered} \tag{2.6}$$

3. Investigation of the stresses $\sigma(\mathbf{n})$ as a function of the elastic constants of the medium is performed conveniently in the local system of coordinates $x_{\alpha'}$ $(\alpha' = 1, 2, 3)$ connected with the normal $\mathbf{n}$ at each point of the surface. We select the following as the local basis

$$\mathbf{e}_3' = \mathbf{n}, \quad \mathbf{e}_1' = \mathbf{n}_0 \times \mathbf{e}_3, \quad \mathbf{e}_2' = \mathbf{n} \times \mathbf{e}_1' \tag{3.1}$$
$$\mathbf{n}_0 = (\sqrt{n_1^2 + n_2^2})^{-1} (n_1\mathbf{e}_1 + n_2\mathbf{e}_2)$$

where $\mathbf{e}_\alpha$ are the unit vectors of the coordinate system connected with the ellipsoid axes, $\mathbf{e}_{\alpha'}$ are the directions of the local system, and $\mathbf{n}_0$ is the normal to the section $n_3 = 0$. In the new basis the stress tensor $\sigma^{\alpha'\beta'}(\mathbf{n})$ along the principal ellipsoid sections will be planar (all the components with subscript 3 equal zero because of the equilibrium conditions). The non-zero components $\sigma^{1'1'}$, $\sigma^{2'2'}$, $\sigma^{1'2'}$ are denoted, respectively, by

$$\sigma^{1'1'} = \sigma_1, \quad \sigma^{2'2'} = \sigma_2, \quad \sigma^{1'2'} = \tau \tag{3.2}$$

We note that the stress σ_1 in the sections $n_1 = 0$ and $n_2 = 0$ is directed perpendicular to the plane of the section, σ_2 is directed along the tangent to the section contour, and conversely in the section $n_3 = 0$. The direction of σ_1 and σ_2 is not defined at the apex of the ellipsoid C; consequently, we agree to consider $\sigma_1(C) = \sigma^{11}(C)$ and $\sigma_2(C) = \sigma^{22}(C)$.

We present expressions for the stresses σ_1, σ_2 and τ along the principal sections of the ellipsoid:

In the section $n_1 = 0$

$$\begin{gathered}\sigma_1 = p_1 [c_{22}n_2^4\sigma^{11}(B) + c_{33}n_3^4\sigma^{11}(C)] + c_{44}^{-1}\eta_1 n_2 n_3 \sigma^{23}(A) + \\ \Delta_{11}^{-1}p_1 n_2^2 n_3^2 \{(kc_{33} - c_{23})(\Delta_{12}\sigma_0^{22} + \Delta_{13}\sigma_0^{33}) + \\ \Delta_{11}[(\Delta_{12}c_{44}^{-1} + c_{13})\sigma_0^{22} + (\Delta_{13}c_{44}^{-1} + c_{12})\sigma_0^{33}] - \\ L(\xi\Delta_{13}k^{-1}\sigma_0^{22} + \xi^{-1}\Delta_{12}\sigma_0^{33})\} \\ \sigma_2 = p_1 [c_{22}n_2^2\sigma^{33}(B) + c_{33}n_3^2\sigma^{22}(C) - \Delta_{11}c_{44}^{-1}n_2 n_3 \sigma^{23}(A)] \\ \tau = \psi_1 [c_{66}^{-1}n_3\sigma^{12}(C) - c_{55}^{-1}n_2\sigma^{13}(B)]\end{gathered} \tag{3.3}$$

In the section $n_2 = 0$

$$\begin{gathered}\sigma_1 = p_2 [c_{11}n_1^4\sigma^{22}(A) + c_{33}n_3^4\sigma^{22}(C)] + c_{55}^{-1}\eta_2 n_1 n_3 \sigma^{13}(B) + \\ \Delta_{11}^{-1}p_2 n_1^2 n_3^2 \{(c_{23} + \Delta_{12}c_{55}^{-1})(\Delta_{11}\sigma_0^{11} + \Delta_{13}\sigma_0^{33}) + \\ [c_{33}(k\Delta_{12} + \Delta_{13}) + \Delta_{11}(\Delta_{22}c_{55}^{-1} - c_{13})]\sigma_0^{22} + \\ c_{33}q_2(\xi l\sigma_0^{22} - k\sigma_0^{33}) - (L\xi^{-1} + c_{22})\Delta_{12}\sigma_0^{33}\} \\ \sigma_2 = p_2 [c_{11}n_1^2\sigma^{33}(A) + c_{33}n_3^2\sigma^{11}(C) - \Delta_{22}c_{55}^{-1}n_1 n_3 \sigma^{13}(B)] \\ \tau = \psi_2 [c_{44}^{-1}n_1\sigma^{23}(A) - c_{66}^{-1}n_3\sigma^{12}(C)]\end{gathered} \tag{3.4}$$

In the section $n_3 = 0$

$$\begin{gathered}\sigma_1 = p_3 [c_{11}n_1^2\sigma^{22}(A) + c_{22}n_2^2\sigma^{11}(B) - c_{66}^{-1}\Delta_{33}n_1 n_2 \sigma^{12}(C)] \\ \sigma_2 = p_3 [c_{11}n_1^4\sigma^{33}(A) + c_{22}n_2^4\sigma^{33}(B)] + c_{66}^{-1}\eta_3 n_1 n_2 \sigma^{12}(C) + \\ \Delta_{11}^{-1}p_3 n_1^2 n_2^2 \{(c_{23} + \Delta_{13}c_{66}^{-1})(\Delta_{11}\sigma_0^{11} + \Delta_{12}\sigma_0^{22}) + \\ [kc_{33}\Delta_{13} + c_{22}\Delta_{12} + \Delta_{11}(\Delta_{33}c_{66}^{-1} - c_{12})]\sigma_0^{33} + \\ kc_{33}q_3(l\xi^{-1}\sigma_0^{33} - \sigma_0^{22}) - (1 - l\xi)c_{33}\Delta_{13}\sigma_0^{22} \\ \tau = \psi_3 [c_{44}^{-1}n_1\sigma^{23}(A) - c_{55}^{-1}n_2\sigma^{13}(B)]\end{gathered} \tag{3.5}$$

The values of p_2, q_2, η_2, ψ_2 and p_3, q_3, η_3, ψ_3 are obtained from p_1, q_1, η_1, ψ_1 in (1.5) by the replacement of the subscripts $1 \leftrightarrow 2$, $4 \leftrightarrow 5$ and $1 \leftrightarrow 3$, $4 \leftrightarrow 6$, respectively.

For an isotropic medium the stresses σ_1, σ_2, τ over the principal sections on the needle surface have the form:

In the section $n_1 = 0$

$$\begin{gathered}\sigma_1 = \sigma_0^{11} + 2\nu(\xi n_3^2 - n_2^2)(\sigma_0^{22} - \xi^{-1}\sigma_0^{33}) - \nu\tau^{23} \\ \sigma_2 = [(1 + 2\xi)n_3^2 - n_2^2]\sigma_0^{22} + [(1 + 2\xi^{-1})n_2^2 - n_3^2]\sigma_0^{33} - \tau^{23} \\ \tau = (1 + \xi)(n_3\sigma_0^{12} - \xi^{-1}n_2\sigma_0^{13}), \quad \tau^{23} = 2\xi^{-1}(1 + \xi)^2 n_2 n_3 \sigma_0^{23}\end{gathered}$$

In the section $n_2 = 0$

$$\sigma_1 = -\nu\varkappa n_1^2\sigma_0^{11} + [1 + 2\xi - \nu\varkappa(1 - 2\nu)n_1^2]\sigma_0^{22} + [\nu(\varkappa + 2\xi^{-1})n_1^2 - 1]\sigma_0^{33} - \nu\tau^{13}$$

$$\sigma_2 = (1 - \varkappa n_1^2)\sigma_0^{11} + [2\nu\xi - \varkappa(1 - 2\nu)n_1^2]\sigma_0^{22} + [(\varkappa + 2\xi^{-1})n_1^2 - 2\nu]\sigma_0^{33} - \tau^{13}$$

$$\tau = (1 + \xi)[(\varkappa\xi)^{-1}(1 + \xi)n_1\sigma_0^{23} - n_3\sigma_0^{12}]$$

$$\tau^{13} = 2\varkappa(1 + \xi^{-1})n_1 n_3 \sigma_0^{13}$$

In the section $n_3 = 0$

$$\sigma_1 = (1 - \varkappa n_1^2)\sigma_0^{11} + [(\varkappa + 2\xi)n_1^2 - 2\nu]\sigma_0^{22} + [2\nu\xi^{-1} - \varkappa(1 - 2\nu)n_1^2]\sigma_0^{33} - \tau^{12}$$

$$\sigma_2 = -\nu\varkappa n_1^2\sigma_0^{11} + [\nu(\varkappa + 2\xi)n_1^2 - 1]\sigma_0^{22} + [1 + 2\xi^{-1} - \nu\varkappa(1 - 2\nu)n_1^2]\sigma_0^{33} - \nu\tau^{12}$$

$$\tau = \xi^{-1}(1 + \xi)[\varkappa^{-1}(1 + \xi)n_1\sigma_0^{23} - n_2\sigma_0^{13}]$$

$$\tau^{12} = 2\varkappa(1 + \xi)n_1 n_2 \sigma_0^{12}, \ \varkappa = (1 - \nu)^{-1}$$

Therefore, explicit expressions in terms of the ellipsoid parameters, the elastic constants of the external medium, and the coordinates of the unit normal vector are obtained for the stresses on the needle surface. Analysis of these expressions shows that the anisotropy of the medium introduces both quantitative and qualitative changes into the behaviour of the stresses.

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A METHOD OF CONSTRUCTING WEIGHTING FUNCTIONS FOR A CIRCULAR CRACK*

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A general formulation is used to consider a static problem for a linearly elastic body with internal circular crack of normal separation. It is shown that the corresponding weighting function enabling a direct calculation to be made of the stress intensity factor (SIF) under arbitrary loading conditions is equal to the product of the axisymmetric weighting function and Poisson's kernel. The known axisymmetric solution /1/ is used to construct, as an example, the weighting function for a circular crack in an unbounded inhomogeneous body with periodic law of variation in the value of Poisson's ratio. An asymptotic analysis of the solution obtained is carried out for a material with rapidly oscillating elastic characteristics.

Some problems of the inhomogeneous theory of elasticity were studied for bodies with variable Poisson's ratio in /1-4/.

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